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PONCELET-GERGONNE CIRCLE, SYMMETRIC POLYNOMIALS AND BARICENTRIC COORDINATES

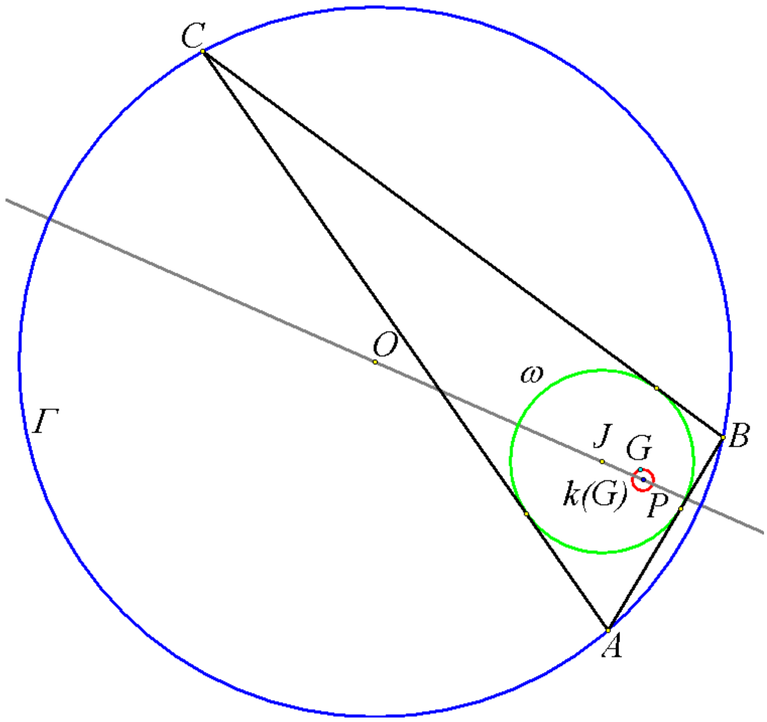
Prof. Sava Grozdev, Prof. Veselin Nenkov, Assoc. Prof. Tatiana Madjarova

Abstract: If two circles are located in a plane in such a way, that one of them is inscribed in a triangle, while the other one is circumscribed with respect to the same triangle, then in case the triangle remains inscribed-circumscribed along its movement between the two circles it is proved by complex numbers in a previous paper that Gergonne point of the triangle describes a circle named Poncelet-Gergonne circle. It is proposed here another approach to the proof using barycentric coordinates with respect to the moving triangle and symmetric polynomials of three variables.
Keywords: triangle, incircle, circumcircle, Gergonne point, Poncelet theorem, symmetric polynomial, Maple

1. Introduction. If two circles $\Gamma(O,R)$ and $\omega(J,r)$ are located in a plane in such a way, that they are the circumcircle and the incircle of a triangle ABC, respectively, then the triangle could move between the two circles remaining inscribed-circumscribed with respect to Γ and ω. In connection with the movement it is proved in [1] the following

Theorem. *If G is Gergonne point of the moving triangle ABC between the circles Γ and ω, then it describes a triangle $k(G)$ with center P on the line OJ in such a way that $OP=\dfrac{4(R+r).OJ}{4R+r}$ and the radius is $\rho=\dfrac{(R-2r)r}{4R+r}$.*

The circle $k(G)$ from the theorem is named Poncelet-Gergonne circle. The theorem itself is proved in [1] by means of complex numbers and applying the computational capabilities of the computer program Maple when preliminary information for the basic parameters of Poncelet-Gergonne circle $k(G)$ are missing. The content of the theorem includes the location of the center and the value of the radius of the circle $k(G)$ which gives us the possibility to apply another idea for its proof. Since the circle $k(G)$ depends on the circles Γ and ω only, we may consider a coordinate system, which is connected with the triangle and moves together with it, while the final numerical result (the radius of the circle) will not depend on the location of the triangle (it depends on the radii R and r of the fixed circles Γ and ω only). For this reason we may consider barycentric coordinates with respect to the moving triangle ABC, namely $A(1,0,0)$ $B(0,1,0)$ and $C(0,0,1)$.



If $BC = a$, $CA = b$ and $AB = c$, the following equalities are satisfied for the coordinates of the points $J(x_J, y_J, z_J)$, $O(x_O, y_O, z_O)$, $G(x_G, y_G, z_G)$ with respect to ΔABC:

(1)
$$x_G = \frac{(a-b+c)(a+b-c)}{(a-b+c)(a+b-c)+(a+b-c)(-a+b+c)+(-a+b+c)(a-b+c)},$$
$$y_G = \frac{(a+b-c)(-a+b+c)}{(a-b+c)(a+b-c)+(a+b-c)(-a+b+c)+(-a+b+c)(a-b+c)},$$
$$z_G = \frac{(-a+b+c)(a-b+c)}{(a-b+c)(a+b-c)+(a+b-c)(-a+b+c)+(-a+b+c)(a-b+c)},$$

(2)
$$x_J = \frac{a}{a+b+c}, \quad y_J = \frac{b}{a+b+c}, \quad z_J = \frac{c}{a+b+c},$$

(3)
$$x_O = \frac{a^2\left(-a^2+b^2+c^2\right)}{16S^2}, \quad y_O = \frac{b^2\left(a^2-b^2+c^2\right)}{16S^2}, \quad z_O = \frac{c^2\left(a^2+b^2-c^2\right)}{16S^2},$$

where

(4)
$$16S^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c).[$$
;

What will be used in the proof of the theorem is the distance between the points $P(x_P, y_P, z_P)$ and $G(x_G, y_G, z_G)$, expressed by the formula

(5) $$PG^2 = -(y_G - y_P)(z_G - z_P)a^2 - (z_G - z_P)(x_G - x_P)b^2 - (x_G - x_P)(y_G - y_P)c^2.$$

It could be noticed from the formulation of the theorem, that the vector equality $\overrightarrow{OP} = k.\overrightarrow{OJ}$ is satisfied, where d$k = \frac{4(R+r)}{4R+r}$. The following equalities are obtained for the coordinates of the center P of the circle $k(G)$:

(6) $$x_P = kx_J + (1-k)x_O, \; y_P = ky_J + (1-k)y_O, \; z_P = kz_J + (1-k)z_O.$$

After substituting (1), (2), (3), (6) in (5) and executing corresponding calculations with Maple we obtain the next result

(7) $$PG^2 = \frac{F}{16S^2\delta^2(4R+r)^2},$$

where

$$F = r^2\xi + 4rR(-a+b+c)(a-b+c)(a+b-c)\eta + 16R^2[(-a+b+c)(a-b+c)(a+b-c)]^2\zeta$$

$$\xi = (a^{10}+b^{10}+c^{10}) - 6(a^9b+ab^9+b^9c+bc^9+c^9a+ca^9) +$$
$$+13(a^8b^2+a^2b^8+b^8c^2+b^2c^8+c^8a^2+c^2a^8) - 8(a^7b^3+a^3b^7+b^7c^3+b^3c^7+c^7a^3+c^3a^7) -$$
$$-14(a^6b^4+a^4b^6+b^6c^4+b^4c^6+c^6a^4+c^4a^6) + 28(a^5b^5+b^5c^5+c^5a^5) +$$
$$+44abc(a^7+b^7+c^7) - 106abc(a^6b+ab^6+b^6c+bc^6+c^6a+ca^6) +$$
$$+114abc(a^5b^2+a^2b^5+b^5c^2+b^2c^5+c^5a^2+c^2a^5) -$$
$$-46abc(a^4b^3+a^3b^4+b^4c^3+b^3c^4+c^4a^3+c^3a^4) + 233a^2b^2c^2(a^4+b^4+c^4) -$$
$$-186a^2b^2c^2(a^3b+ab^3+b^3c+bc^3+c^3a+ca^3) + 92a^2b^2c^2(a^2b^2+b^2c^2+c^2a^2) +$$
$$+80a^3b^3c^3(a+b+c)$$

$$\eta = 2(a^7+b^7+c^7) - 7(a^6b+ab^6+b^6c+bc^6+c^6a+ca^6) +$$
$$+9(a^5b^2+a^2b^5+b^5c^2+b^2c^5+c^5a^2+c^2a^5) - 4(a^4b^3+a^3b^4+b^4c^3+b^3c^4+c^4a^3+c^3a^4) +$$
$$+28abc(a^4+b^4+c^4) - 25abc(a^3b+ab^3+b^3c+bc^3+c^3a+ca^3) +$$
$$+8abc(a^2b^2+b^2c^2+c^2a^2) + 16a^2b^2c^2(a+b+c)$$

$$\zeta = (a^4+b^4+c^4) - (a^3b+ab^3+b^3c+bc^3+c^3a+ca^3) + abc(a+b+c)$$

$$\delta = 2ab+2bc+2ca-a^2-b^2-c^2$$

Introduce the notations $\sigma_1 = a+b+c$, $\sigma_2 = ab+bc+ca$, $\sigma_3 = abc$. The following equalities are known from the triangle geometry

(8) $$\sigma_1 = 2p, \ \sigma_2 = p^2+r^2+4Rr, \ \sigma_3 = 4Rrp.$$

Using the equalities (8) and applying some results for symmetric polynomials that are described in [2], we obtain the following equalities by means of the program product Maple:

$$a^2+b^2+c^2 = \sigma_1^2 - 2\sigma_2 = 2(p^2-4Rr-r^2),$$

$$a^2b^2+b^2c^2+c^2a^2 = \sigma_2^2 - 2\sigma_1\sigma_3 = p^4 - 2r(4R-r)p^2 + r^2(4R+r)^2,$$

$$a^3b+ab^3+b^3c+bc^3+c^3a+ca^3 = \sigma_1^2\sigma_2 - 2\sigma_2^2 - \sigma_1\sigma_3 = 2p^4 - 8Rrp^2 - 32R^2r^2 - 16Rr^3 - 2r^4,$$

$$a^4+b^4+c^4 = \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 + 4\sigma_1\sigma_3 = 2(p^2-2rp-4Rr-r^2)(p^2+2rp-4Rr-r^2),$$

$$a^4b^3+a^3b^4+b^4c^3+b^3c^4+c^4a^3+c^3a^4 = \sigma_1\sigma_2^3 - 3\sigma_1^2\sigma_2\sigma_3 - \sigma_2^2\sigma_3 + 5\sigma_1\sigma_3^2 =$$
$$= 2p\left[p^6 - r(14R-3r)p^4 + r^2(16R^2-4Rr+3r^2)p^2 + r^3(2R+r)(4R+r)^2\right],$$

$$a^5b^2+a^2b^5+b^5c^2+b^2c^5+c^5a^2+c^2a^5 = \sigma_1^3\sigma_2^2 - 2\sigma_1^4\sigma_3 - 3\sigma_1\sigma_2^3 + 6\sigma_1^2\sigma_2\sigma_3 + 3\sigma_2^2\sigma_3 - 7\sigma_1\sigma_3^2 =$$
$$= 2p\left[p^6 - r(14R+r)p^4 + r^2(48R^2+20Rr-5r^2)p^2 - 3r^3(2R+r)(4R+r)^2\right],$$

$$a^6b+ab^6+b^6c+bc^6+c^6a+ca^6 = \sigma_1^5\sigma_2 - 5\sigma_1^3\sigma_2^2 - \sigma_1^4\sigma_3 + 5\sigma_1\sigma_2^3 + 9\sigma_1^2\sigma_2\sigma_3 - 7\sigma_2^2\sigma_3 - 4\sigma_1\sigma_3^2 =$$

$$=2p\left[p^6-r(10R+9r)p^4+r^2(32R^2+4Rr-5r^2)p^2+r^3(6R+5r)(4R+r)^2\right],$$

$$a^7+b^7+c^7=\sigma_1^7-7\sigma_1^5\sigma_2+14\sigma_1^3\sigma_2^2+7\sigma_1^4\sigma_3-7\sigma_1\sigma_2^3-21\sigma_1^2\sigma_2\sigma_3+7\sigma_2^2\sigma_3+7\sigma_1\sigma_3^2=$$
$$=2p\left[p^6-7r(2R+3r)p^4+7r^2(16R^2+20Rr+5r^2)p^2-7r^3(2R+r)(4R+r)^2\right],$$

$$a^5b^5+b^5c^5+c^5a^5=\sigma_2^5-5\sigma_1\sigma_2^3\sigma_3+5\sigma_1^2\sigma_3^2+5\sigma_2^2\sigma_3^2-5\sigma_1\sigma_3^3=$$
$$=p^{10}-5r(4R-r)p^8+10r^2(8R^2-4Rr+r^2)p^6+10r^6p^4+5r^6(4R+r)^2p^2+r^5(4R+r)^5,$$

$$a^6b^4+a^4b^6+b^6c^4+b^4c^6+c^6a^4+c^4a^6=$$
$$=\sigma_1^2\sigma_2^4-4\sigma_1^3\sigma_2^2\sigma_3+2\sigma_1^4\sigma_3^2-2\sigma_2^5+8\sigma_1\sigma_2^3\sigma_3-9\sigma_2^2\sigma_3^2+2\sigma_1\sigma_3^3=$$
$$=2p^{10}-2r(20R-3r)p^8+4r^2(44R^2-8Rr++r^2)p^6-4r^3(8R^2-r^2)(4R-r)p^4+$$
$$+2r^4(8R^2+8Rr-3r^2)(4R+r)^2p^2-2r^5(4R+r)^5,$$

$$a^7b^3+a^3b^7+b^7c^3+b^3c^7+c^7a^3+c^3a^7=\sigma_1^4\sigma_2^3-3\sigma_1^5\sigma_2\sigma_3-4\sigma_1^2\sigma_2^4+12\sigma_1^3\sigma_2^2\sigma_3+$$
$$+3\sigma_1^4\sigma_3^2+2\sigma_2^5-2\sigma_1\sigma_2^3\sigma_3-24\sigma_1^2\sigma_2\sigma_3^2+6\sigma_2^2\sigma_3^2+11\sigma_1\sigma_3^3=$$
$$=2p^{10}-2r(20R+3r)p^8+28r^2(8R^2+4Rr-r^2)p^6-4r^3(96R^3+7r^3)p^4-$$
$$-2r^4(32R^2+32Rr+3r^2)(4R+r)^2p^2+2r^5(4R+r)^5,$$

$$a^8b^2+a^2b^8+b^8c^2+b^2c^8+c^8a^2+c^2a^8=\sigma_1^6\sigma_2^2-2\sigma_1^7\sigma_3-6\sigma_1^4\sigma_2^3+12\sigma_1^5\sigma_2\sigma_3+$$
$$+9\sigma_1^2\sigma_2^4-12\sigma_1^3\sigma_2^2\sigma_3-13\sigma_1^4\sigma_3^2-2\sigma_2^5-8\sigma_1\sigma_2^3\sigma_3+28\sigma_1^2\sigma_2\sigma_3^2+\sigma_2^2\sigma_3^2-10\sigma_1\sigma_3^3=$$
$$=2p^{10}-2r(20R+13r)p^8+4r^2(88Rr+84R^2-7r^2)p^6-$$
$$-4r^3(352R^3+360R^2r+60Rr^2-7r^3)p^4+2r^4(88R^2+72Rr+13r^2)(4R+r)^2p^2-$$
$$-2r^5(4R+r)^5,$$

$$a^9b+ab^9+b^9c+bc^9+c^9a+ca^9=\sigma_1^8\sigma_2-8\sigma_1^6\sigma_2^2-\sigma_1^7\sigma_3+20\sigma_1^4\sigma_2^3+15\sigma_1^5\sigma_2\sigma_3-$$
$$-16\sigma_1^2\sigma_2^4-46\sigma_1^3\sigma_2^2\sigma_3-7\sigma_1^4\sigma_3^2+2\sigma_2^5+31\sigma_1\sigma_2^3\sigma_3+33\sigma_1^2\sigma_2\sigma_3^2-15\sigma_2^2\sigma_3^2-7\sigma_1\sigma_3^3=$$
$$=2p^{10}-2r(16R+27r)p^8+4r^2(76R^2+98Rr+21r^2)p^6-$$
$$-4r^3(160R^3+40R^2r-70Rr^2-21r^3)p^4-2r^4(56R^2+92Rr+27r^2)(4R+r)^2p^2+$$
$$+2r^5(4R+r)^5,$$

$$a^{10}+b^{10}+c^{10}=\sigma_1^{10}-10\sigma_1^8\sigma_2+35\sigma_1^6\sigma_2^2+10\sigma_1^7\sigma_3-50\sigma_1^4\sigma_2^3-60\sigma_1^5\sigma_2\sigma_3+25\sigma_1^2\sigma_2^4+$$
$$+100\sigma_1^3\sigma_2^2\sigma_3+25\sigma_1^4\sigma_3^2-2\sigma_2^5-40\sigma_1\sigma_2^3\sigma_3-60\sigma_1^2\sigma_2\sigma_3^2+15\sigma_2^2\sigma_3^2+10\sigma_1\sigma_3^3=$$
$$=2p^{10}-10r(4R+9r)p^8+140r^2(2R+3r)(2R+r)p^6-$$
$$-20r^3(160R^3+280R^2r+140Rr^2+21r^3)p^4+10r^4(40R^2+40Rr+9r^2)(4R+r)^2p^2-$$
$$-2r^5(4R+r)^5.$$

Substitute the above equalities in the expressions ξ, η, ζ and δ in the corresponding places. Then

$$\xi=256r^4p^2\left[12Rrp^2+(9R^2-24Rr+4r^2)(4R+r)^2\right],$$
$$\eta=128Rr^4p\left[3(4R-r)p^2-(6R-5r)(4R+r)^2\right],$$

$\zeta = -64R^2r^2\left[3p^2 - (4R+r)^2\right],$

$\delta = 4\sigma_2 - \sigma_1^2 = 4r(4R+r).$

From here and the dependence

$$(-a+b+c)(a-b+c)(a+b-c) = \frac{16S^2}{a+b+c} = \frac{16p^2r^2}{2p} = 8r^2p$$

we conclude the expression for F:

$$F = 256p^2r^6(R-2r)^2(4R+r)^2.$$

Further, substitute the expressions F and δ in (7) obtaining $PG^2 = \frac{(R-2r)^2 r^2}{(4R+r)^2}$, i.e.

$$\rho = \frac{(R-2r)r}{4R+r}.$$

This ends the proof of the theorem.

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Prof. Sava Grozdev, DSc
Researcher ID: AAG-4146-2020
ORCID 0000-0002-1748-7324
Faculty of Mathematics and Informatics
Plovdiv University "Paisii Hilendarski"
236, Blvd. Bulgaria
10, Saint Ekaterina Street
4027 Plovdiv, Bulgaria
E-mail: sava.grozdev@gmail.com

Prof. Veselin Nenkov, PhD
Researcher ID: AAB-5776-2019
Nikola Vaptsarov Naval Acedamy
73, Vasil Drumev Street
9002 Varna, Bulgaria
E-mail: v.nenkov@nvna.eu

Assoc. Prof. Tatiana Madjarova, PhD
Nikola Vaptsarov Naval Acedamy
73, Vasil Drumev Street
9002 Varna, Bulgaria
E-mail: tmadjarova@nvna.eu